Uitwerkingen Tentamen Complexe Analyse Jan. 31, 2012 1.

1\. a) The Cauchy Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ become:

$$3x^2+3y^2-3 = 3y^2+3x^2-3 \quad (1)$$
$$6xy = -(6xy) \quad (2)$$

To show that f is differentiable on the coordinate axes, I will apply Theorem 5, page 74. Let $z_0 = x_0+iy_0$

$\boxed{8}$ with $x_0=0 \vee y_0=0$. Take any open set G containing z_0. The first partial derivatives exist in G, are continuous in G, and, finally, the CR-equations hold at z_0. Hence $f(z)$ differentiable at z_0!

b) Suppose $z_0 \in \mathbb{C}$ and f analytic at z_0. By definition then there is an open neighbourhood G of z_0 on

$\boxed{8}$ which f is differentiable. However every such neighbourhood contains at least one point $z_1 = x_1+iy_1$ with both $x_1 \neq 0$ and $y_1 \neq 0$. There the CR-equations fail to hold, so $f(z)$ not differentiable there!!

2\. a) The function $g(z)$ is analytic on $\mathbb{C}\setminus\{1,-1\}$. The contour Γ_1 is a closed contour in this domain

$\boxed{6}$ and therefore $\int_{\Gamma_1} g(z)\,dz = 0$ by Cauchy's integral theorem.

b) $g(z)$ has isolated singularities $z=1$ and $z=-1$ in the interior of the contour Γ_2, but is further analytic inside and on Γ_2. The residue theorem then says $\int_{\Gamma_2} g(z)\,dz = 2\pi i\left[\operatorname{Res}(z=1) + \operatorname{Res}(z=-1)\right]$

Now,

$\boxed{12}$

$$\operatorname{Res}(z=1) = \lim_{z\to 1}(z-1)\frac{e^z}{z^2-1} = \lim_{z\to 1}\frac{e^z}{z+1} = \frac{1}{2}e$$

$$\operatorname{Re}(z=-1) = \lim_{z\to -1}(z+1)\frac{e^z}{z^2-1} = \lim_{z\to -1}\frac{e^z}{z-1} = -\frac{1}{2}\frac{1}{e}$$

Hence $\int_{\Gamma_2} g(z)\,dz = \pi i\,(e - 1/e)$.

3. a) We have $\cos w = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \frac{w^6}{6!} + \dots$
Hence $\cos\frac{1}{z} = 1 - \frac{1}{2!\,z^2} + \frac{1}{4!\,z^4} - \frac{1}{6!\,z^6} + \dots$
This yields the Laurent series at $z=0$:

$\boxed{8}$

$$z^2\cos\frac{1}{z} = z^2 - \frac{1}{2!} + \frac{1}{4!\,z^2} - \frac{1}{6!\,z^4} + \dots$$

b) The residue of $f(z)$ at $z=0$ is equal to the coefficient of $\frac{1}{z}$ in the Laurent series, this is obviously equal to 0, so $\mathrm{Res}(f,0)=0$.

$\boxed{4}$

c) $z=0$ is an essential singularity since the Laurent series at 0 has infinitely many negative powers of z (this is how essential singularities are defined)

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4. a) "Candidates" are: $z=0$, $z=k\pi$, $k=\pm1,\pm2,\dots$
Taylor series of $\sin z$ at $z=k\pi$: $(k\neq 0)$:
k even: $\sin z = z-k\pi - \frac{1}{3!}(z-k\pi)^3 + \frac{1}{5!}(z-k\pi)^5 - \dots$
k odd: $\sin z = -(z-k\pi) + \frac{1}{3!}(z-k\pi)^3 - \frac{1}{5!}(z-k\pi)^5 + \dots$
Define in both cases

$$g(z) := (z-k\pi)\frac{ze^{iz}}{\sin z}$$

$\boxed{12}$

Then $$g(z) = \frac{ze^{iz}}{1 \pm \frac{1}{3!}(z-k\pi)^2 \pm \frac{1}{5!}(z-k\pi)^4 \pm \dots}$$

where the denominator is an analytic function, non-zero at $z=k\pi$. Hence $g(z)$ is analytic at $k\pi$ and we have in a neighbourhood of $k\pi$:

$$\frac{ze^{iz}}{\sin z} = \frac{g(z)}{z-k\pi}, \quad g(k\pi)\neq 0$$

By Lemma 7, page 281, $f(z)$ has a pole of order 1 at $z=k\pi$.

b) Clearly $\lim_{z \to 0} \frac{z}{\sin z} = \lim_{z\to 0} \frac{1}{\frac{\sin z}{z}} = 1$

$\boxed{6}$ So $z=0$ is a removable singularity of $f(z)$

c) After defining $f(0)=1$, the new function is analytic in the open disc $|z| < \pi$ (since the "first" poles appear at $z = \pm\pi$)

$\boxed{6}$ By theorem 3, page 243, $f(z)$ has a Taylor Series on this open disc, and the radius of convergence is equal to π

5. a) If $f(z)$ and $h(z)$ are analytic inside and on the simple closed contour C, and the strict

$\boxed{5}$ inequality $|h(z)| < |f(z)|$ holds for every $z \in C$ then f and $f+h$ have the same number of zeros inside C

b) Define $f(z) = 6z^4$ and $h(z) = z^3 - 2z^2 + z - 1$. Let C be the contour $|z|=1$. On the contour C we have

$$|h(z)| \leq |z^3| + |2z^2| + |z| + 1$$

$\boxed{6}$
$$= |z|^3 + 2|z|^2 + |z| + 1 = 5$$

$$|f(z)| = |6z^4| = 6|z|^4 = 6$$

Since $|h(z)| < |f(z)|$ there, $g(z) = f(z) + h(z)$ and $f(z)$ both have 4 zeros inside C.

c) Define $f(z) = z$, $h(z) = 1$ and C the contour $|z|=1$. On the contour we have $|h(z)| = 1 \leq |f(z)|$

$\boxed{5}$ However, $f(z)$ has 1 zero $(z=0)$ inside C, while $f(z) + h(z) = z+1$ has no zeros inside C